

H W 1

i. Consider the pattern $\dots A \underset{-1}{A} \underset{0}{A} \underset{1}{A} \underset{2}{A} A \dots$

The group of symmetries is made up of the following elements:

- $T_n, n \in \mathbb{Z}$, where T_n is the pattern translation that moves the A in the " 0 " slot to the A in the " n " slot. (Note that T_0 is the identity.)
- $R_n, R_{n+y_2}, n \in \mathbb{Z}$ where R_n is the pattern reflection around a vertical line passing through the " n " slot and R_{n+y_2} is the pattern reflection around a vertical line passing between the " n " slot and the " $n+1$ " slot.

The group is not abelian. Note that, for example,

$$R_0 T_1 (\dots A \boxed{A} A \dots) = \dots A A A \dots$$

$$\quad \quad \quad -1 \quad 0 \quad 1 \quad \quad \quad -1 \quad 0 \quad 1$$

$$T_1 R_0 (\dots A \boxed{A} A \dots) = \dots A A \boxed{A} \dots$$

$$\quad \quad \quad -1 \quad 0 \quad 1 \quad \quad \quad -1 \quad 0 \quad 1$$

Thus, $R_0 T_1 \neq T_1 R_0$.

2. Symmetries for figures in 1.22 (arranged in the same grid):

$$\{R_0, R_{90}, R_{180}, R_{270}\}$$

$$D_5$$

$$\{R_0, D_{90}, S_{180}\}$$

$$\{R_0, R_{180}\}$$

$$D_4$$

$$\{R_0, R_{180}, R_{240}\}$$

$$\{D_3\}$$

$$D_{16}$$

$$D_7$$

$$D_4$$

$$D_5$$

$$\{R_0, R_{36}, R_{72}, R_{108}, R_{144}\}$$

3. $U(10) = \{1, 3, 7, 9\}$. Its Cayley table is

	1	3	7	9
1	1	3	7	9
3	3	9	1	7
7	7	1	9	3
9	9	7	3	1

4. Claim: $\{5, 15, 25, 35\}$, multiplication mod 40, is a group.

Proof Consider the multiplication table:

	5	15	25	35
5	25	35	5	15
15	35	25	15	5
25	5	15	25	35
35	15	5	35	25

From the table, we can see that the set is closed under the operation. Also, 25 is an identity since $25 \cdot a = a$ for $a = 5, 15, 25, 35$. To show inverses, we note that $a^2 = 25$ for $a = 5, 15, 25, 35$ so that each element is its own inverse. Since arithmetic mod 40 is associative, this shows the set is a group.

5. Let $G = \left\{ \begin{bmatrix} 1 & a & b \\ 0 & 1 & c \\ 0 & 0 & 1 \end{bmatrix} \mid a, b, c \in \mathbb{R} \right\}$, matrix multiplication.

To show G is a group, we first check closure.

Let $A = \begin{bmatrix} 1 & a_1 & b_1 \\ 0 & 1 & c_1 \\ 0 & 0 & 1 \end{bmatrix}$, $B = \begin{bmatrix} 1 & a_2 & b_2 \\ 0 & 1 & c_2 \\ 0 & 0 & 1 \end{bmatrix} \in G$.

Then, $A \cdot B = \begin{bmatrix} 1 & a_1 + a_2 & b_1 + a_1 c_2 + b_2 \\ 0 & 1 & c_1 + c_2 \\ 0 & 0 & 1 \end{bmatrix}$. Since $a_1 + a_2, b_1 + a_1 c_2 + b_2$ and $c_1 + c_2$ are all real numbers, we see that $A \cdot B \in G$.

Note that $\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \in G$ since $a = b = c = 0$ are real numbers and this is an identity for matrix multiplication.

To check inverses, note that for $A = \begin{bmatrix} 1 & a & b \\ 0 & 1 & c \\ 0 & 0 & 1 \end{bmatrix}$, if

we set $B = \begin{bmatrix} 1 & -a & -b+ac \\ 0 & 1 & -c \\ 0 & 0 & 1 \end{bmatrix}$, then $B \in G$ and $AB = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$.

Since matrix multiplication is associative, G is a group. \square

6. Let G be a group such that for any $a, b, c \in G$, if $ab = ca$, then $b = c$. Then G is abelian.

Proof] Let $x, y \in G$. We want to show $xy = yx$.

To do this, set $a = x$, $b = yx$, and $c = xy$.

Then,

$$\begin{aligned} (xy)x &= (xy)x \\ &= x(yx) \quad \text{by the associative} \\ &\quad \text{property.} \end{aligned}$$

$$\text{or } ca = ab$$

Using the cross-cancellation property, this means

$b = c$, or equivalently, $xy = yx$

\square

7. Suppose G is a group and $a \in G$ such that $a^{10} = e$. Then $|a| = 1, 2, 5$, or 10 .

Proof] First note that if $|a| = 1, 2, 5$, or 10 , then $a^{10} = e$.

(For example, if $|a| = 2$, then $a^{10} = (a^2)^5 = e^5 = e$.)

To prove these are the only possibilities, suppose

$|a| = i$ for some $i \neq 1, 2, 5$, or 10 . Note that i must be less than 10 and we can write $10 = k + ni$ for some $0 < k < i$. Then $e = a^{10} = (a^i)^n \cdot a^k = a^k$. Since $k < i$, this contradicts $|a| = i$. □

8. Suppose G is abelian and contains ≥ 2 elements of order 2. Then G has a subgroup of order 2.

4.

Proof] Let $a, b \in G$ w/ $|a| = 2$, $|b| = 2$, and $a \neq b$.

First observe that $ab \neq e$, a, b since
• $ab = e$ would imply $b = a^{-1}$. However, we know
 $a^{-1} = a \neq b$.

• $ab = a$ would imply $b = e$, contradicting $|b| = 2$.

• $ab = b$ would imply $a = e$, contradicting $|a| = 2$.

Therefore $\{e, a, b, ab\}$ is a set of size 4.

To show it is a subgroup, by Theorem 3.3

it's enough to show it is closed.

Below is the multiplication table for this set:

	e	a	b	ab
e	e	a	b	ab
a	a	$a^2 = e$	ab	$a^2b = b$
b	b	$ba = ab$	$b^2 = e$	$bab = ab^2 = a$
ab	ab	$aba = a^2be = ab = a$	$ab^2 = a$	$(ab)^2 = e = b$

(Note we are using the fact that G is abelian in the calculations, i.e. $ab = ba$.)

The table shows the set is closed and thus it is a subgroup of order 4.